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# Mathematics: analysis and approaches

## Higher level

### Paper 3

9 May 2023

Zone A afternoon | Zone B morning | Zone C afternoon

1 hour

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#### Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 25]

**In this question, you will be investigating the family of functions of the form  $f(x) = x^n e^{-x}$ .**

Consider the family of functions  $f_n(x) = x^n e^{-x}$ , where  $x \geq 0$  and  $n \in \mathbb{Z}^+$ .

When  $n = 1$ , the function  $f_1(x) = x e^{-x}$ , where  $x \geq 0$ .

(a) Sketch the graph of  $y = f_1(x)$ , stating the coordinates of the local maximum point. [4]

(b) Show that the area of the region bounded by the graph  $y = f_1(x)$ , the  $x$ -axis and the line  $x = b$ , where  $b > 0$ , is given by  $\frac{e^b - b - 1}{e^b}$ . [6]

You may assume that the total area,  $A_n$ , of the region between the graph  $y = f_n(x)$  and the  $x$ -axis can be written as  $A_n = \int_0^\infty f_n(x) dx$  and is given by  $\lim_{b \rightarrow \infty} \int_0^b f_n(x) dx$ .

(c) (i) Use l'Hôpital's rule to find  $\lim_{b \rightarrow \infty} \frac{e^b - b - 1}{e^b}$ . You may assume that the condition for applying l'Hôpital's rule has been met. [2]

(ii) Hence write down the value of  $A_1$ . [1]

You are given that  $A_2 = 2$  and  $A_3 = 6$ .

(d) Use your graphic display calculator, and an appropriate value for the upper limit, to determine the value of

(i)  $A_4$ ; [2]

(ii)  $A_5$ . [1]

(e) Suggest an expression for  $A_n$  in terms of  $n$ , where  $n \in \mathbb{Z}^+$ . [1]

(f) Use mathematical induction to prove your conjecture from part (e). You may assume that, for any value of  $m$ ,  $\lim_{x \rightarrow \infty} x^m e^{-x} = 0$ . [8]

Turn over

2. [Maximum mark: 30]

**In this question, you will investigate the maximum product of positive real numbers with a given sum.**

Consider the two numbers  $x_1, x_2 \in \mathbb{R}^+$ , such that  $x_1 + x_2 = 12$ .

(a) Find the product of  $x_1$  and  $x_2$  as a function,  $f$ , of  $x_1$  only. [2]

(b) (i) Find the value of  $x_1$  for which the function is maximum. [1]

(ii) Hence show that the maximum product of  $x_1$  and  $x_2$  is 36. [1]

Consider  $M_n(S)$  to be the maximum product of  $n$  positive real numbers with a sum of  $S$ , where  $n \in \mathbb{Z}^+$  and  $S \in \mathbb{R}^+$ .

For  $n = 2$ , the maximum product can be expressed as  $M_2(S) = \left(\frac{S}{2}\right)^2$ .

(c) Verify that  $M_2(S) = \left(\frac{S}{2}\right)^2$  is true for  $S = 12$ . [1]

Consider  $n$  positive real numbers,  $x_1, x_2, \dots, x_n$ .

The geometric mean is defined as  $(x_1 \times x_2 \times \dots \times x_n)^{\frac{1}{n}}$ . It is given that the geometric mean is always less than or equal to the arithmetic mean, so  $(x_1 \times x_2 \times \dots \times x_n)^{\frac{1}{n}} \leq \frac{(x_1 + x_2 + \dots + x_n)}{n}$ .

(d) (i) Show that the geometric mean and arithmetic mean are equal when  $x_1 = x_2 = \dots = x_n$ . [2]

(ii) Use this result to prove that  $M_n(S) = \left(\frac{S}{n}\right)^n$ . [4]

(e) Hence determine the value of

(i)  $M_3(12)$ ; [1]

(ii)  $M_4(12)$ ; [1]

(iii)  $M_5(12)$ . [1]

For  $n \in \mathbb{Z}^+$ , let  $P(S)$  denote the maximum value of  $M_n(S)$  across all possible values of  $n$ .

(f) Write down the value of  $P(12)$  and the value of  $n$  at which it occurs. [2]

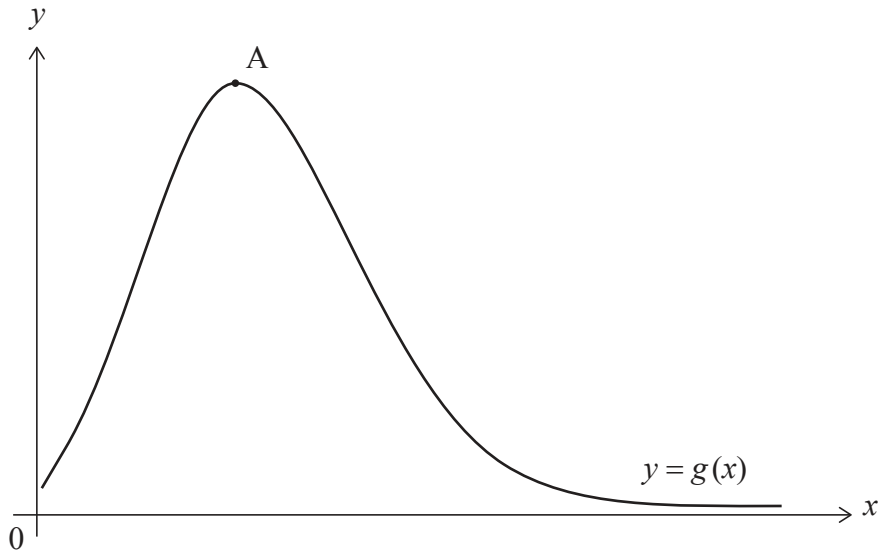
(g) Determine the value of  $P(20)$  and the value of  $n$  at which it occurs. [3]

**(This question continues on the following page)**

**(Question 2 continued)**

Consider the function  $g$ , defined by  $\ln(g(x)) = x \ln\left(\frac{S}{x}\right)$ , where  $x \in \mathbb{R}^+$ .

A sketch of the graph of  $y = g(x)$  is shown in the following diagram. Point A is the maximum point on this graph.



- (h) Find, in terms of  $S$ , the  $x$ -coordinate of point A. [6]
- (i) Verify that  $g(x) = M_x(S)$ , when  $x \in \mathbb{Z}^+$ . [2]
- (j) Use your answer to part (h) to find the largest possible product of positive numbers whose sum is 100. Give your answer in the form  $a \times 10^k$ , where  $1 \leq a < 10$  and  $k \in \mathbb{Z}^+$ . [3]

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**References:**

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